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Modified Black Hole Thermodynamics with Generalized Uncertainty Principle

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Abstract In this paper we propose a way of determining the subleading corrections to the Bekenstein-Hawking black hole entropy by considering a modified generalized uncertainty principle with two parameters. In the context of modified generalized uncertainty principle, coefficients of the correction terms of black hole entropy are written in terms of combination of the parameters. We also obtained the corrections to the Stefan-Boltzman law and the black hole evaporation in terms of the parameters. By estimating those parameters, say by experiment, one can test results from other context of quantum gravity theories such as black hole entropy.

Keywords Generalized uncertainty principle · Black hole thermodynamics

1 Introduction

The calculation of the black hole entropy is the first test of all quantum gravity theories. The leading term in the black hole entropy is the Bekenstein-Hawking (BH) entropy formula, which says that the entropy is proportional to the horizon area A of the black hole, i.e. $S_{BH} = A/4G$ [1–3].

In general, one expects that the black hole entropy is a function of the horizon area A due to holography [4, 5] and the quantum corrected black hole entropy would take the following extensive form [6]:

$$S = \frac{A}{4L_P^2} + c_0 \ln\left(\frac{A}{4L_P^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4L_P^2}\right)^{-n} + \text{const}, \tag{1}$$

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where the coefficients $\{c_n\}$ can be regarded as model dependent parameters. Indeed known quantum gravity theories, e.g. string theory and loop-quantum gravity, have obtained the above form of black hole entropy in their own context [7, 8]. Recently, a non-relativistic renormalizable theory of gravity at a Lifshitz point has been proposed by Hořava, namely Hořava-Lifshitz (HL) gravity which reduces to general relativity at large scales [9]. We note that the entropy for black holes in the deformed HL gravity with a special value of coupling constant also contains a logarithmic term [10].

However it seems hard to have an agreement on matching subleading terms to the black hole entropy. In this regard we will consider the way of determining the subleading correction terms of the black hole entropy in the context of Generalized Uncertainty Principle (GUP) which is typically written in the literature as follows:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha L_P^2 \frac{\Delta p}{\hbar}, \quad L_P = \sqrt{\frac{G\hbar}{c^3}} \tag{2}$$

where L_p is Planck length.

One can say that GUP is a gravitational generalization of Heisenberg uncertainty principle. Noticing $L_P^2/\hbar$ can be replaced by the Newton's constant G, we can be aware that the correction term is a consequence of gravity. It first appeared in the context of string theory [11–14], which has a fundamental length scale that leads to the intrinsic limitation of resolution on probing processes using strings. We can also obtain GUP by considering gravitational interaction between quantum particles like electron and photon in which the gravitational effect is no longer small enough to ignore [15]. Then the right hand side of the standard uncertainty relation $\Delta x \geq \hbar/\Delta p$ gains a correction proportional to Δp which gives rise to minimum value of the position uncertainty, $2\sqrt{\alpha}L_P$.

It is known that the black hole entropy can be understood by considering uncertainty relation. One way of understanding this procedure is as follows [16]: we may regard a radius of the black hole horizon as an intrinsic uncertainty in the position of an emitting photon near the horizon. Identifying the energy uncertainty ($\Delta E = \Delta pc$) of the photon with the characteristic energy E of the emitting photon, we can translate ΔE as the Hawking temperature T. (Here, we set the Boltzman constant $k_B = 1$.) Then we can compute the black hole entropy from the first thermodynamic law, $dS = T^{-1}dE$. It agrees with the Bekenstein-Hawking entropy, $A/4L_P^2$, up to a calibration factor 2π. Concerning GUP instead of the standard uncertainty relation, Adler et al. obtained modified Hawking temperature and black hole entropy with a logarithmic correction term [17].

This is a very interesting result because it shows that features of gravity theories such as black hole entropy can be seen generically by means of uncertainty principle which is model independent. Furthermore, one can obtain corrections in all orders to the entropy with the coefficients depending on only a single parameter α in GUP (9) as follows [18]:

$$S = \frac{A}{4L_P^2} - \frac{\pi\alpha^2}{4}\ln\left(\frac{A}{4L_P^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4L_P^2}\right)^{-n} + \text{const}, \tag{3}$$

where the expansion coefficients $c_n \sim \alpha^{2(n+1)}$. It means that we can fix the coefficients of the subleading terms of the black hole entropy by determining α in experiment. Then we can compare the coefficients with ones obtained from other gravity theories. The above form of quantum corrected entropy of the Schwartzschild black hole from GUP can also be related to the entropy of black holes in the deformed Hořava-Lifshitz gravity [19, 20].

A single parameter, however, will not be enough to obtain the precise corrections to the entropy. We note that for matching beyond logarithmic subleading correction terms we need more generalized form of GUP with additional parameters. For this, we will consider GUP with one more parameter β written as

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha L_P^2 \frac{\Delta p}{\hbar} + \beta L_P^4 \frac{(\Delta p)^3}{\hbar^3}. \tag{4}$$

We will call this generalized GUP as GUP*. By considering GUP* we will show that β plays the role of fixing the coefficient of the correction term to the black hole entropy in $1/A$. In this regard, we expect that we need at least one more parameter to fix the coefficient of the next order correction term to the black hole entropy.

In addition, we will see a modification of Stefan-Boltzman law by GUP*. If one can detect the energy spectrum precisely, say by experiments, one can fix α and β in GUP*. We also point out that precision measurements of hydrogen-atom spectrum [21], the Lamb shift, Landau levels, and the current in a Scanning Tunnel Microscope(STM) give other ways of testing GUP [22, 23].

Quite recently, there has been following progress in the direction related to our interests. Thermodynamics of black hole in the context of generalized uncertainty principle has been more intensively discussed in Refs. [24, 25]. A way of determining the minimal length parameter from the black hole entropy-area relationship is proposed in Ref. [26]. The GUP-corrected Bekentein-Hawking entropy in the higher dimensional spacetime is considered in Ref. [27].

We organize our paper as follows: In Sect. 2, we introduce the modified generalized uncertainty principle (GUP*) with higher order terms. Then we calculate the black hole entropy from GUP* and the Hawking Temperature in Sect. 3. We see that the Stefan-Boltzman law has corrections in terms of α and β in Sect. 4. Finally in Sect. 5, we conclude with some comments.

2 Generalized Uncertainty Principle

In string theory, due to the fundamental length scale of the string l_s, there is a restriction in probing smaller distance than l_s generically. It suggests that the Heisenberg's uncertainty principle in quantum mechanics should be modified in the regime in which physics is governed by string theory as follows:

$$\Delta x \Delta p \geq \hbar + a l_s^2 (\Delta p)^2, \tag{5}$$

where a is a constant. Note that the minimum bound of Δx proportional to l_s is encoded in the right hand side of the relation [11].

This relation can also be obtained when we consider a gravitational interaction between quantum particles. As shown in Ref. [15], in a process of probing an electron by a photon in a region with size L, concerning a photon as a classical particle with effective mass, $m = E/c^2$, the electron will experience an acceleration

$$\frac{d^2 t}{dr^2} = -\frac{G(E/c^2)}{r^2}\hat{r}, \tag{6}$$

where r is the distance between the electron and the photon. Assuming that the interaction occurs in a characteristic time L/c, and the photon-electron distance is of order $r \approx L$, the distance of which the electron moves during the interaction time will be obtained as

$$\Delta x_G \approx \frac{GE}{c^2L^2}\left(\frac{L}{c}\right)^2 = \frac{Gp}{c^3}. \tag{7}$$

Here, we introduced the photon momentum using $E = pc$. Noting that the electron momentum uncertainty will be of order of the photon momentum, we have

$$\Delta x_G \approx \frac{G\Delta p}{c^3} = L_P^2\frac{\Delta p}{\hbar}, \quad L_P = \sqrt{\frac{G\hbar}{c^3}}, \tag{8}$$

where L_P is the Planck length. Then we have modified uncertainty relation by adding (8) to the Heisenberg's uncertainty relation as

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha L_P^2\frac{\Delta p}{\hbar}, \tag{9}$$

where α is a dimensionless constant.[1] We call this relation as Generalized Uncertainty Principle (GUP). Note that Δx has a minimum bound $2\sqrt{\alpha}L_P$. Recall that (5) derived in string theory by considering string scattering has similar form with GUP. In this respect, l_s is comparable to L_P.

In general ground, position uncertainty relation may have more correction terms in Δp as we consider a high energy microscopic collision process between quantum particles in Planckian regime. In this point of view, we will consider the more generalized uncertainty principle with higher oder correction terms. For this, we start by taking following form of GUP given in Ref. [28]:

$$\Delta x \geq \frac{1}{\Delta p} + \alpha L_P^2\Delta p + O\left(L_P^3(\Delta p)^2\right). \tag{10}$$

Here, we have set $c = \hbar = 1$. In this paper, we will focus on GUP with higher order terms up to of order $(\Delta p)^3$ introducing new parameters other than α as follows:

$$\Delta x \geq \frac{1}{\Delta p} + \alpha L_P^2\Delta p + \gamma L_P^3(\Delta p)^2 + \beta L_P^4(\Delta p)^3 + \cdots. \tag{11}$$

We shall call this GUP* in what follows. Subsequently, we will discuss black hole entropy and the Hawking radiation from GUP*.

3 GUP* and Black Hole Physics

In following discussion about the black hole thermodynamics, let us begin by considering a way of re-expressing GUP* in Δp for convenience. In the case of GUP of (9), we obtain the exact form of Δp expression as follows:

$$\Delta p \geq \frac{\Delta x}{2\alpha L_P^2}\left(1 - \sqrt{1 - \frac{4\alpha L_P^2}{(\Delta x)^2}}\right). \tag{12}$$

[1] This derivation of GUP based on Newtonian theory is very heuristic but still sufficient to understand the modification of the uncertainty relation due to gravity. Further derivation based on general relativity is also given in Ref. [15].

Obtaining this expression, we took a negative choice of sign to agree with classical result in the limit $L_P \to 0$. Since GUP gives rise to minimum value of the position uncertainty, $(\Delta x)_{\min} = 2\sqrt{\alpha}L_P$, for the case of $\Delta x > 2\sqrt{\alpha}L_P$, we can expand the square root part in (12) and have [18]

$$\Delta p \geq \frac{1}{\Delta x} + \frac{\alpha L_P^2}{(\Delta x)^3} + \frac{2\alpha^2 L_P^4}{(\Delta x)^5} + \cdots. \tag{13}$$

In turn, let us consider GUP* of (11). In order to obtain an exact expression of Δp of GUP* in terms of Δx, we need to solve a quartic equation in Δp. However, since we are interested in an expansion form of Δp, we deal with this problem using a series solution method considering L_P as an expansion parameter. Concerning the coefficients of higher order terms with β and γ in (11) are relatively small, we expect that the Δp expression of GUP* can be written in the vicinity of (13) as follows:

$$\begin{aligned}\Delta p &\geq p_0 + \sum_{n=1}^{\infty} p_n L_P^n \\ &= \frac{1}{\Delta x} + p_1 L_P + \left(\frac{\alpha}{(\Delta x)^3} + p_2\right) L_P^2 + p_3 L_P^3 + \left(\frac{2\alpha^2}{(\Delta x)^5} + p_4\right) L_P^4 + \cdots,\end{aligned} \tag{14}$$

where p_0 is the right hand side of (13). Substituting this into (11), we can read off the values of p_n as

$$p_1 = 0, \quad p_2 = 0, \quad p_3 = \frac{\gamma}{(\Delta x)^4}, \quad p_4 = \frac{\beta}{(\Delta x)^5}, \cdots \tag{15}$$

and we finally have expansive form of Δp for GUP* as

$$\Delta p \geq \frac{1}{\Delta x} + \frac{\alpha L_P^2}{(\Delta x)^3} + \frac{\gamma L_P^3}{(\Delta x)^4} + \frac{2\alpha^2 + \beta}{(\Delta x)^5} L_P^4 + \cdots. \tag{16}$$

It has been known that if we consider a photon as a probe of quantum particle with position uncertainty Δx in relativistic case, the energy of the quantum particle have lower bound as $E \geq 1/\Delta x$ [29]. Thus, we have

$$E \geq \frac{1}{\Delta x} + \frac{\alpha L_P^2}{(\Delta x)^3} + \frac{\gamma L_P^3}{(\Delta x)^4} + \frac{2\alpha^2 + \beta}{(\Delta x)^5} L_P^4 + \cdots. \tag{17}$$

Now, let us consider the black hole entropy. For this, we take an analysis given by Bekenstein who noticed that any process of assimilation of a quantum particle by a black holes necessarily leads to increase of horizon area [2]. Thus the nondecreasing entropy can be identified with the nondecreasing black hole horizon area. In this analysis, the quantum particle is considered to have a finite proper radius b while its center of mass follows classical trajectory. The minimum increase of horizon area occurs if the particle is captured a proper distance b away from the horizon, that is, when the particle's center of mass is at a turning point. For the case when neutral particle is captured, we have

$$\Delta A \geq 8\pi L_P^2 \mu b, \tag{18}$$

where μ is a rest mass of the particle and the horizon radius is set $r_{\text{Sch}} = 2L_P^2 M$. For a point particle, $b = 0$ and it results in $\Delta A \geq 0$ [30]. However, according to Bekenstein's original observation, a relativistic quantum particle cannot be localized better than its Compton wave length. Thus, b cannot be taken smaller than its uncertainty Δx. Therefore, concerning μ as an energy of the particle E, one finds a lower bound on increase of the black hole horizon area as

$$(\Delta A)_{\min} = 8\pi L_P^2 E \Delta x. \tag{19}$$

Here we will closely follow Bekenstein's analysis. Let us now consider the area bound applying GUP*. Substituting (17) to (19), and introducing a constant parameter ϵ, we have

$$\Delta A \geq \epsilon L_P^2 \left(1 + \frac{\alpha L_P^2}{(\Delta x)^2} + \frac{\gamma L_P^3}{(\Delta x)^3} + \frac{2\alpha^2 + \beta}{(\Delta x)^4} L_P^4 + \cdots\right). \tag{20}$$

We note that constant ϵ may be different valued from one in (19) when we consider a charged particle assimilation by a charged black holes [31, 32]. Concerning the minimal increase of entropy as one "bit" of information which is typically known as ln2 [30], we can relate the area with entropy as follows:

$$\begin{aligned} \frac{dS}{dA} \simeq \frac{(\Delta S)_{\min}}{(\Delta A)_{\min}} &\simeq \frac{\ln 2}{\epsilon L_P^2 (1 + \frac{\alpha L_P^2}{(\Delta x)^2} + \frac{\gamma L_P^3}{(\Delta x)^3} + \frac{2\alpha^2+\beta}{(\Delta x)^4} L_P^4 + \cdots)} \\ &\simeq \frac{\ln 2}{\epsilon L_P^2}\left(1 - \frac{\alpha L_P^2}{(\Delta x)^2} - \frac{\gamma L_P^3}{(\Delta x)^3} - \frac{\alpha^2 + \beta}{(\Delta x)^4} L_P^4 + \cdots\right). \end{aligned} \tag{21}$$

Now we should take some length scale which corresponds to Δx. As discussed in Ref. [18], for the particles near the horizon of Schwarzschild black hole, their Compton length is of order of the inverse of the Hawking temperature (or inverse of the surface gravity) proportional to r_{Sch}. Therefore, taking r_{Sch} would be a proper choice of length scale for Δx. Related argument can be also found in Refs. [17, 33]. Based on these observations, let us choose $\Delta x \sim 2r_{\mathrm{Sch}}$. It leads to $(\Delta x)^2 \sim A/\pi$ and we can rewrite (21) as

$$\frac{dS}{dA} \simeq \frac{\ln 2}{\epsilon L_P^2}\left(1 - \frac{\alpha\pi L_P^2}{A} - \frac{\gamma \pi^{\frac{3}{2}} L_P^3}{A^{3/2}} - \frac{(\alpha^2 + \beta)\pi^2 L_P^4}{A^2} + \cdots\right). \tag{22}$$

By integrating above equation, and setting $\epsilon = 4\ln 2$, we obtain the black hole entropy with corrections to the Bekenstein-Hawking entropy:

$$S = \frac{A}{4L_P^2} - \left(\frac{\pi\alpha}{4}\right) \ln\left(\frac{A}{4L_P^2}\right) + \sum_{n=1}^{\infty} c_{\frac{n}{2}} \left(\frac{A}{4L_P^2}\right)^{-\frac{n}{2}} + \text{const.} \tag{23}$$

where c_n are written as

$$\begin{aligned} c_{\frac{1}{2}} &= \left(\frac{\pi}{4}\right)^{\frac{3}{2}} 2\gamma \\ c_1 &= \left(\frac{\pi}{4}\right)^2 (\alpha^2 + \beta) \\ c_{\frac{3}{2}} &= \left(\frac{\pi}{4}\right)^{\frac{5}{2}} 2\alpha\gamma \\ c_2 &= \left(\frac{\pi}{4}\right)^3 (\alpha^2 + 2\alpha\beta + \gamma^2) \cdots . \end{aligned} \tag{24}$$

Let us analyze this result. Recall that we obtained the black hole entropy from GUP* of (11) which is taken by arbitrarily generalizing GUP. As a result, we see that the entropy of (23) contains correction terms in of order $1/\sqrt{A}$. Tracing back the origin of the coefficients $c_{\frac{1}{2}}$ and $c_{\frac{3}{2}}$, one can find that these correction terms appeared due to $\gamma L_P^3(\Delta p)^2$ correction term in GUP*. In fact, these correction terms look rather unusual. As far as we know, there has been no known gravity theory in which the black hole entropy has correction terms in of order $1/\sqrt{A}$. On the other hand, from the view point of GUP*, there is no reason to exclude the $\gamma L_P^3(\Delta p)^2$ term. It would be interesting if we find some models which correspond to these new terms.

In this paper, we will proceed our analysis with GUP* which leads to the comparable form of black hole entropy with one given in (1) by setting γ be zero;

$$\Delta x \geq \frac{1}{\Delta p} + \alpha L_P^2 \Delta p + \beta L_P^4 (\Delta p)^3. \tag{25}$$

Then the minimum value of the position uncertainty is given as follows:

$$(\Delta x)_{\min} = \sqrt{\frac{2}{27\beta}} \left(-\alpha + \sqrt{\alpha^2 + 12\beta}\right)^{1/2} \left(2\alpha + \sqrt{\alpha^2 + 12\beta}\right) L_P. \tag{26}$$

Here, the parameters α and β are regarded as positive constants. This minimum value is greater than one from GUP, $2\sqrt{\alpha} L_P$. It also suggests the minimum size of the black hole horizon:

$$(\Delta x)_{\min} \sim 2(r_{\mathrm{Sch}})_{\min}. \tag{27}$$

Since the horizon radius of the Schwarzschild black hole is determined by a black hole mass, one can expect a minimum mass for the black hole from above relation [17, 28, 34]. If we take into account the horizon radius as $r_{\mathrm{Sch}} = 2L_P^2 M$, the minimum mass can be written as follows:[2]

$$M_{\min} = \frac{1}{12}\sqrt{\frac{2}{3\beta}} \left(-\alpha + \sqrt{\alpha^2 + 12\beta}\right)^{1/2} \left(2\alpha + \sqrt{\alpha^2 + 12\beta}\right) M_P, \tag{28}$$

where $M_P = 1/L_P$.

We can now write the black hole entropy obtained from GUP* given in (25) as

$$S = \frac{A}{4L_P^2} - \left(\frac{\pi\alpha}{4}\right) \ln\left(\frac{A}{4L_P^2}\right) + \sum_{n=1}^{\infty} c_n \left(\frac{A}{4L_P^2}\right)^{-n} + \text{constant}, \tag{29}$$

where c_n are given as follows:

$$c_1 = \left(\frac{\pi}{4}\right)^2 (\alpha^2 + \beta), \qquad c_2 = \left(\frac{\pi}{4}\right)^3 (\alpha^3 + 2\alpha\beta), \ldots. \tag{30}$$

[2] We have assumed a linear relation between the horizon radius and the mass of the black hole, since a precise relation between them is not known for GUP corrected black holes. However, we still expect that there will be a minimum mass for the black hole.

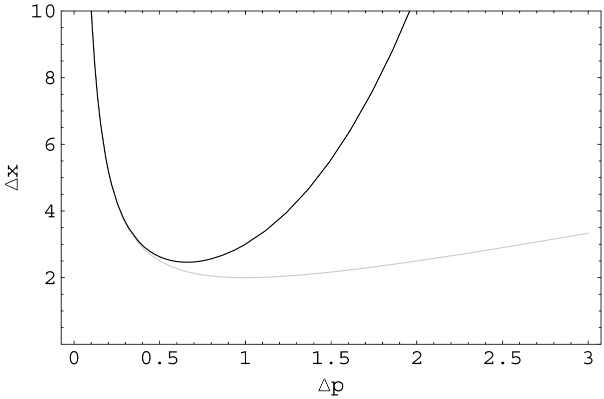

Fig. 1 The uncertainty in position in units of Planck length versus the uncertainty in momentum in units of Planck mass. The *gray line* denotes GUP result with $\alpha = 1$, $\beta = 0$ and the *solid line* denotes GUP* result with $\alpha = 1$, $\beta = 1$

Now let us take a limit of $\alpha \to 0$ in GUP*. It is the L_P^4 correction to the standard uncertainty principle written as follows:

$$\Delta x \geq \frac{1}{\Delta p} + \beta L_P^4 (\Delta p)^3. \tag{31}$$

Eliminating α from (29), we have the final form of the entropy as

$$\begin{aligned} S &= \frac{A}{4L_P^2} + \beta\left(\frac{\pi}{4}\right)^2\left(\frac{A}{4L_P^2}\right)^{-1} + \beta^2\left(\frac{\pi}{4}\right)^4\left(\frac{A}{4L_P^2}\right)^{-3} + \cdots \\ &= \frac{A}{4L_P^2} + \sum_{n=1}^{\infty}\left(\frac{\beta\pi^2}{16}\right)^n\left(\frac{A}{4L_P^2}\right)^{-2n+1} + \text{constant}. \end{aligned} \tag{32}$$

Note that GUP with the pure $(\Delta p)^3$ ($\alpha = 0$) term does not give rise to a logarithmic term. We get only odd power terms in $1/A$. It means that while β plays a crucial role in fixing the second correction term of the black hole entropy, the logarithmic correction term is determined only by α. we also expect that we need L_P^6 corrected GUP with a $(\Delta p)^5$ term for fixing a coefficient of third order correction term of the entropy.

Now one can also consider the Hawking temperature of the black hole using first thermodynamic law, $dM = TdS$. Taking into account the black hole area as $A = 16\pi M^2 L_P^4$, and using (29), the temperature $T_{\mathrm{GUP*}}$ can be obtained as follows:

$$T_{\mathrm{GUP*}} = \frac{{M_P}^2}{8\pi M}\left(1 + \frac{\alpha {M_P}^2}{16M^2} + \frac{(2\alpha^2 + \beta){M_P}^4}{256M^4} + \cdots\right). \tag{33}$$

The leading term shows the standard Hawking temperature of the Schwarzschild black hole, and others are correction terms affected by GUP*. We note that $T_{\mathrm{GUP*}}$ has a maximum value when M reaches to minimum mass $M_{\min}$. Figure 2 represents the temperature of the black hole versus the black hole mass for each case of standard uncertainty principle and GUP*.

4 Black Hole Evaporation

In this section, we will consider the black hole evaporation. Some earlier works on modification of the black hole spectrum can be found in Refs. [17, 28]. As discussed in Ref. [28],

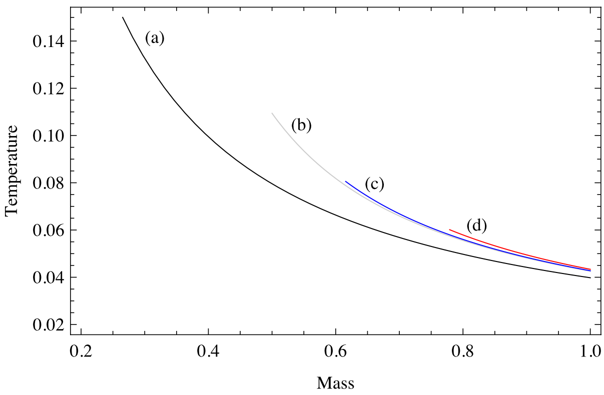

Fig. 2 Temperature of a black hole versus the mass in units of Planck mass. (*a*) A result of standard uncertainty with $\alpha = 0$ and $\beta = 0$; (*b*) A GUP result with $\alpha = 1$, $\beta = 0$; (*c*) A GUP* result for $\alpha = 1$, $\beta = 1$; (*d*) A GUP* result for $\alpha = 1$, $\beta = 5$

when we consider a GUP, one obtains a modified black body spectrum. To see this, we assume that the de Broglie wave length is modified by GUP* and takes a form as follows:

$$\lambda \simeq \frac{2\pi}{p}(1 + \alpha L_P^2 p^2 + \beta L_P^4 p^4). \tag{34}$$

Inverting this relation with $E = p$ and $\omega = \frac{2\pi}{\lambda}$, we get following expression of E:

$$E \simeq \omega(1 + \alpha L_P^2 \omega^2 + (2\alpha^2 + \beta) L_P^4 \omega^4), \tag{35}$$

up to second order in α and first order in β. We note that we take this approximation for the rest of our analysis.

Let us first consider the black body radiation of photons in a box with edges of length L. The boundary condition for the wave lengths of the photons are given as $\frac{1}{\lambda} = \frac{n}{2L}$, where n is an integer number. Thus number of modes of the photons in an infinitesimal angular frequency interval $[\omega, \omega + d\omega]$ is given by

$$g(\omega)d\omega = \frac{V}{\pi^2}\omega^2 d\omega \tag{36}$$

where $V = L^3$. In Bose-Einstein statistics, the average energy of the photons per oscillator, $\bar{E}$, can be obtained as

$$\bar{E} = \frac{E}{e^{\frac{E}{T}} - 1}. \tag{37}$$

Since the average energy will be influenced by GUP*, using (35), we have

$$\begin{aligned}\bar{E} \approx \frac{\omega}{e^{\frac{\omega}{T}} - 1}\Bigg[&1 + (1 - F)\alpha L_P^2 \omega^2 \\ &+ \left\{\left(2 - F\left(3 + \frac{\omega}{2T}\right) + F^2\right)\alpha^2 + (1 - F)\beta\right\} L_P^4 \omega^4\Bigg],\end{aligned} \tag{38}$$

where

$$F \equiv 1 - \frac{\omega/T}{1 - e^{-\frac{\omega}{T}}}. \tag{39}$$

Then the modified Stefan-Boltzman law takes the form

$$
\begin{aligned}
u(T) &= \frac{1}{V}\int_0^{\infty} \bar{E} g(\omega) d\omega \\
&\approx \frac{\pi^2}{15}T^4 - \frac{40\pi^4\alpha}{63}L_P^2 T^6 + \frac{56\pi^6(2\alpha^2-\beta)}{15}L_P^4 T^8,
\end{aligned}
\tag{40}
$$

where $u(T)$ is the energy density per unit volume.

To deal with the black hole evaporation from (40), we will consider the radiation emitted by a black hole as a black body radiation. Using the energy conservation equation,

$$
\frac{dM}{dt} = -Au(T_{\mathrm{GUP}^*}),
\tag{41}
$$

we have

$$
\frac{dM}{dt} = -\frac{M_P^4}{3840\pi M^2}\left(1 + \frac{17\alpha M_P^2}{168M^2} + \frac{(94\alpha^2+7\beta)M_P^4}{3584M^4}\right).
\tag{42}
$$

One can rewrite it introducing $m = M/M_P$ and the characteristic time, $t_{\mathrm{ch}} = 3840\pi T_P$, where $T_P = 1/M_P$ is the Planck time as follows:

$$
\frac{dm}{dt} = -\frac{1}{t_{\mathrm{ch}}m^2}\left(1 + \frac{17\alpha}{168m^2} + \frac{94\alpha^2+7\beta}{3584m^4}\right).
\tag{43}
$$

Integrating this equation, replacing a variable m to x, we can obtain a relation between evaporating time and black hole mass as

$$
\frac{t}{t_{ch}} \approx \left[\frac{x^3}{3} - \frac{17\alpha x}{168} + \left(\frac{1805\alpha^2}{112896} + \frac{\beta}{512}\right)\frac{1}{x}\right]_{m_f}^{m_i},
\tag{44}
$$

where m_i is the initial mass of the black hole in units of Planck mass. The first term of the right hand side represents a result from the standard uncertainty relation which says that the black hole evaporates to zero mass in finite time. For our case, however, since the black hole cannot be smaller than M_{min}, the evaporation stops when the black hole mass reaches to M_{min} at time

$$
\frac{t}{t_{\mathrm{ch}}} = \frac{63}{192}m^3 - \frac{17\alpha}{224}m - \left(\frac{1805\alpha^2}{37632} + \frac{3\beta}{512}\right)\frac{1}{m}.
\tag{45}
$$

This is a refined result of GUP by considering GUP* with one more parameter β. In this procedure GUP* prevents the black hole from vanishing by evaporation. One can see that the decay time of the black hole from GUP* is shorter than one from GUP.

5 Discussion

In this work we have considered a generalized uncertainty principle with higher order correction term to the ordinary GUP with one more dimensionless parameter β. Considering L_P^4 corrected GUP as given in (25) with dimensionless parameters α and β, we obtained quantum corrected black hole entropy. It turns out that the coefficients of correction terms of black hole entropy are given by combinations of the parameters in GUP*.

Moreover with these two parameters, we can determine exactly up to second order correction terms of the entropy. GUP* also leads modification of Stefan-Boltzman law for the Hawking radiation. Therefore if we can detect the radiation rate of the black hole precisely, we can fix the values of the parameters α and β. By comparing known entropy in various other theories to the entropy from our trial uncertainty relation, one can test the quantum gravity in remarkably simple way.

One interesting direction of applying the result from GUP* is what we may use it for obtaining information of extra dimensions appeared in string theory by experiment. The quantum corrections to the black hole entropy with a logarithmic term has been obtained in string theory [35]. In this case, the coefficients of the correction terms are written in terms of topological quantities of a six dimensional Calabi-Yau manifold which is considered as a geometry of compactified extra dimensions. By comparing the coefficients of correction terms to the black hole entropy in GUP* to the ones obtained in string theory, one can determine the value of topological quantities such as Hodge numbers which specify the extra dimension. However, since the black hole considered in string theory is an extremal black hole (more precisely a BPS black hole), while the GUP procedure is based on considering non-extremal black holes, it seems hard to compare the entropy obtained in both contexts for now. Nevertheless, as we have progress on dealing with microscopic nature of black holes in string theory, we expect that string theory may be able to extend their analysis to a class of non-extremal black holes. Then it would be an interesting trial to apply our idea for those black holes.

The other challenging problem is what Heisenberg algebra gives this kind of GUP. For GUP, corresponding Heisenberg algebra has been known as κ-deformed type which leads to noncommutative space time [36–38] but for GUP*, the mathematical structure of the algebra would be different.

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